Total number of printed pages–8

14 (MAT–1) 1036 (N/O)

2021

(Held in 2022)

MATHEMATICS

Paper : MAT–1036

(Mechanics and Tensor)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

FIRST HALF

(Mechanics)

Marks : 50

(New and Old Course)

1. Answer ***any one*** : 10

   (a) A particle moves on a smooth sphere under no forces except the pressure of the surface ; show that its path is given by the equation $\cot\theta = \cot\beta\cos\phi$, where θ, ϕ are its angular coordinates.

Contd.

(b) A heavy particle of mass m moves on the smooth inner surface of a sphere of radius a, and its greatest and least depth below the centre are $\frac{a}{2}$ and $\frac{a}{4}$. Show that when the depth below the centre is z, the normal reaction is

$$3mg\,\frac{z+\frac{a}{2}}{a}.$$

2. Answer ***any one*** of the following: 8

(a) A rectangular lamina, whose sides are of length $2a$ and $2b$, is at rest when one corner is caught and suddenly made to move with prescribed speed V in the plane of the lamina. Show by using Kelvin's theorem that the greater angular velocity which can thus be imparted to the lamina is $\frac{3V}{4\sqrt{a^2+b^2}}$.

(b) A circular disc of mass M and radius r, has a rod AB smoothly linked to a point on the circumference so that AB, when produced passes through the centre of the disc. The rod and the disc lie on a smooth horizontal table A blow P is given to the rod at the extreme A at right angles to it, prove that the K.E. produced is $\frac{9P^2}{10M}$, $2M$ being the mass of the rod.

3. Answer ***any two*** parts : 6×2=12

(a) A rigid body with one point fixed rotates with angular velocity $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$. With usual notations, show that the K. E. of the body at time t is given by

$$T = \frac{1}{2}\left[A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2D\omega_2\omega_3 - 2E\omega_3\omega_1 - 2F\omega_1\omega_2\right]$$

(b) A rectangular lamina of sides $2a$ and $2b$ rotating about a diagonal with constant angular velocity $\vec{\omega}$. Use Euler's equation to determine the components of the moment of the forces about the principal axes of the lamina.

(c) A uniform square lamina is rotating under no impressed forces about one corner O, which is fixed. If $\omega_1, \omega_2, \omega_3$ are the components of angular velocity about the principal axes of O (in the order of increasing moments of inertia), show that

$\omega_1^2 + \omega_2^2$ and $3\omega_2^2 + 4\omega_3^2$ are constants.

(d) If T is the total K.E. of rotation of a rigid body with one point fixed, prove that $\frac{dT}{dt} = \vec{\omega} \cdot \dot{\vec{\Lambda}}$, where all the quantities refer to the principal axes of the body.

4. Answer ***any two*** parts : 6×2=12

(a) With usual notations, derive Lagrange's equations for conservative field of forces in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha} \quad ;$$

$\alpha = 1, 2, 3 \cdots,$ where

$L = T - V.$

(b) Use Lagrange's equation to discuss the motion of a particle in a plane under an attractive central force obeying the law of inverse square.

(c) Obtain the equation of motion for the Lagrangian

$$L = a^2 (1 - \cos\theta)\, \dot{\theta}^2 - ag\,(1 + \cos\theta)$$

5. Answer ***any one***: 8

(a) Write the Hamiltonian function for the motion of a compound pendulum and hence obtain the equation of motion for a compound pendulum.

(b) Define the Hamiltonian and obtain Hamilton's equations in the form

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \ \dot{q}_\alpha = +\frac{\partial H}{\partial p_\alpha}$$

$(\alpha = 1, 2, 3, \cdots, n)$, where the symbols have their usual meanings.

SECOND HALF

(Tensor)

Marks : 30

(New Course)

1. Answer ***any three*** parts of the following : 5×3=15

(a) What is meant by contraction of tensors? Show that the process of contraction reduces the rank of a tensor by two.

(b) (i) Show that the inner product of a controvariant vector and a covariant vector is a scalar invariant.

(ii) Define symmetric tensor. Show that the symmetric property of tensors remains unchanged due to the transformation of coordinates.

(c) Prove the following :

(i) $\Gamma_{ij,k} + \Gamma_{jk,i} = \dfrac{\partial g_{ki}}{\partial x^j}$

(ii) $\Gamma^i_{ij} = \dfrac{1}{2g}\dfrac{\partial g}{\partial x^j}$

(d) Establish the transformation law of the Christoffel symbol of the second kind.

2. Answer ***any three*** parts of the following : 5×3=15

(a) Derive the x^k-covariant derivative of the covariant tensor A_{ij} of rank two with respect to the fundamental tensor g_{ij}.

(b) (i) Find the covariant derivative of a scalar invariant function.

(ii) Show that the gradient of a scalar invariant function ϕ is a covariant vector.

(c) (i) Define divergence of a vector and show that $div\, A_i = div\, A^i$.

(ii) Define *curl* of a vector and show that $curl\ A_i = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$

(d) Deduce the expression

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial \phi}{\partial x^j} \right)$$

where the symbols have their usual meanings.

(Old Course)

1. Answer ***any three*** of the following : 5×3=15

   (a) Define symmetric and anti-symmetric tensors. Show that every tensor of rank two can be expressed as a sum of two tensors, one of which is symmetric and the other anti-symmetric.

   (b) State and prove the Quotient Theorem of tensors.

   (c) Define the function g^{ij} suitably and prove that it is a symmetric contravariant tensor of rank two. Hence show that $g_{ij}g^{jk} = \delta_i^k$.

   (d) Establish the transformation law of the Christoffel symbol of the second kind.

2. Answer ***any three*** of the following : 5×3=15

   (a) Derive the x^j-covariant derivative of the contravariant vector A^i with respect to the fundamental tensor g_{ij}.

   (b) Show that the fundamental tensors are covariant constant.

   (c) If A_{ij} is the *curl* of a covariant vector, prove that $A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$.

   (d) Define Laplacian of a scalar invariant function and express it in tensor form.
