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3 (Sem–6/CBCS) MAT HC 1 (N/O)

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–6016

(New Syllabus/Old Syllabus)

Full Marks : 80/60

Time : Three hours

The figures in the margin indicate full marks for the questions.

New Syllabus

Full Marks : 80

(Riemann Integration and Metric Spaces)

1. Answer the following as directed : $1 \times 10 = 10$

   (a) Define the discrete metric d on a non-empty set X.

Contd.

(b) Let F_1 and F_2 be two subsets of a metric space (X, d). Then

(i) $\overline{F_1 \cup F_2} = \overline{F_1} \cap \overline{F_2}$

(ii) $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$

(iii) $\overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$

(iv) $\overline{F_1 \cap F_2} = \overline{F_1} \cup \overline{F_2}$

(Choose the correct option)

(c) Let (X, d) be a metric space and $A \subset X$. Then

(i) $Int\,A$ is the largest open set contained in A.

(ii) $Int\,A$ is the largest open set containing A.

(iii) $Int\,A$ is the intersection of all open sets contained in A.

(iv) $Int\,A = A$

(Choose the correct option)

(d) Let (X, d) be a disconnected metric space.

We have the statements :

I. There exists two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

II. There exists two non-empty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(i) Only I is true

(ii) Only II is true

(iii) Both I and II are true

(iv) None of I and II is true

(Choose the correct option)

(e) Find the limit points of the set of rational numbers Q in the usual metric R_u.

(f) In a metric space, the intersection of infinite number of open sets need not be open. Justify it with an example.

(g) Define a mapping $f : X \to Y$, so that the metric spaces $X = [0, 1]$ and $Y = [0, 2]$ with usual absolute value metric are homeomorphic.

(h) Define Riemann sum of f for the tagged partition (P, t).

(i) State the first fundamental theorem of calculus.

(j) Examine the existence of improper Riemann integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

2. Answer the following questions : 2×5=10

(a) Prove that in a metric space (X, d) every open ball is an open set.

(b) Prove that the function $f:[0,1]\to R$ defined by $f(x)=x^2$ is an uniformly continuous mapping.

(c) Let d_1 and d_2 be two matrices on a non-empty set X. Prove that they are equivalent if there exists a constant K such that

$$\frac{1}{K} d_2(x, y) \le d_1(x, y) \le K d_2(x, y)$$

(d) If m is a positive integer, prove that $\overline{|m+1} = m!$

(e) Let $f(x) = x$ on $[0, 1]$.

Let $P = \left\{ x_i = \frac{i}{4}, \ i = 0, \dots 4 \right\}$

Find $L(f, P)$ and $U(f, P)$.

3. Answer the following questions ***(any four)***: 5×4=20

(a) Let (X, d) be metric space and F be a subset of X. Prove the F is closed in X if and only if F^c is open.

(b) Define diameter of a non-empty bounded subset of a metric space (X, d). If A is a subset of a metric space (X, d), then prove that $d(A) = d(\overline{A})$. 1+4=5

(c) Let (X, d) be a metric space. Then prove that the following statements are equivalent :

(i) (X, d) is disconnected.

(ii) There exists two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

(d) Let $f, g : [a, b] \to R$ be integrable functions. Then prove that $f+g$ is integrable and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(e) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p}\,dx$ for various values of p.

(f) Consider $f : [0, 1] \to R$ defined by $f(x) = x^2$. Prove that f is integrable.

4. Answer the following questions : 10×4=40

(a) (i) Let X be the set of all bounded sequences of numbers $\{x_i\}_{i \geq 1}$ such that $\sup_i |x_i| < \infty$.

For $x = \{x_i\}_{i \geq 1}$ and $y = \{y_i\}_{i \geq 1}$ in X define $d(x, y) = \sup_i |x_i - y_i|$.

Prove that d is a metric on X. 5

(ii) Prove that a convergent sequence in a metric space is a Cauchy sequence. Is the converse true? Justify with an example. 4+1=5

Or

(a) (i) Show that $d(x, y) = \sqrt{|x-y|}$ defines a metric on the set of reals. 4

(ii) Show that the metric space $X = \mathbb{R}^n$ with the metric given by

$$d_p(x, y) = \left(\Sigma |x_i - y_i|^p\right)^{1/p}, \quad p \geq 1$$

where $x = (x_1, x_2 \ldots\ldots x_n)$ and $y = (y_1, y_2 \ldots\ldots y_n)$ are in $\mathbb{R}^n$ is a complete metric space. 6

(b) (i) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$. If f is continuous on X, prove the following : 3+3=6

(i) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all subsets of B of Y

(ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X

(ii) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be uniformly continuous. Prove that if $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in X, then $\{f(x_n)\}_{n \geq 1}$ is a Cauchy sequence in Y. 4

Or

(b) Define fixed point of a mapping $T: X \to X$. Let $T: X \to X$ be a contraction of the complete metric space (X, d). Prove that T has a unique fixed point. 2+8=10

(c) *(i)* Prove that if the metric space (X, d) is disconnected, then there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0). 5

(ii) Let (X, d) be a metric space and A^0, B^0 are interiors of the subsets A and B respectively. Prove that

$$(A \cap B)^0 = A^0 \cap B^0;$$

$$(A \cup B)^0 \supseteq A^0 \cup B^0.$$ 5

Or

(c) *(i)* When is a non-empty subset Y of a metric space (X, d) said to be connected? Let (X, d_X) be a connected metric space and $f:(X, d_X) \to (Y, d_Y)$ be a continuous mapping. Prove that the space $f(X)$ with the metric induced from Y is connected. 5

(ii) Let (X, d) be a metric space and $Y \subseteq X$. If X is separable then prove that Y with the induced metric is also separable. 5

(d) (i) If f is Riemann integrable on $[a, b]$ then prove that it is bounded on $[a, b]$. 5

(ii) When is an improper Riemann integral said to exist?
Show that the improper integral of $f(x) = |x|^{-1/2}$ exists on $[-1, 1]$ and its value is 4. 1+4=5

Or

(d) (i) Let $f : [a, b] \to R$ be integrable. Then prove that the indefinite integral $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$.

Further prove that if f is continuous at $x \in [a, b]$, then F is differentiable at x and $F'(x) = f(x)$. 3+3=6

(ii) Evaluate

$$\lim_{x \to \infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{\sqrt{n^3}} = \frac{2}{3}$$ 4

Old Syllabus

Full Marks : 60

(*Complex Analysis*)

1. Answer the following as directed : $1 \times 7 = 7$

(a) Any complex number $z = (x, y)$ can be written as

(i) $z = (0, x) + (1, 0)(0, y)$

(ii) $z = (x, 0) + (0, 1)(y, 0)$

(iii) $z = (x, 0) + (0, 1)(0, y)$

(iv) $z = (0, x) + (1, 0)(y, 0)$

(Choose the correct option)

(b) Write the function $f(z) = z^2 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

(c) The value of $\lim_{z \to \infty} \frac{2z + i}{z + 1}$ is

(i) ∞

(ii) 0

(iii) 2

(iv) i

(Choose the correct option)

(d) Determine the singular points of the function

$$f(z) = \frac{z^2 + 1}{(z+2)(z^2 + 2z + 2)}.$$

(e) Define an analytic function of the complex variable *z*.

(f) $e^{i(2n+1)\pi}$ is equal to

(i) 1

(ii) −1

(iii) 0

(iv) 2

(Choose the correct option)

(g) $Log(-1)$ is equal to

(i) $\frac{\pi}{2}i$

(ii) πi

(iii) $-\frac{\pi}{2}i$

(iv) $-\pi i$

(Choose the correct option)

2. Answer the following questions : 2×4=8

(a) Show that $\lim_{z \to \infty} \frac{1+z^2}{z-1} = \infty$

(b) If $f(z) = e^x \cdot e^{iy} = e^z$ where $z = x + iy$, show that $f'(z) = e^x \cos y + i e^x \sin y$.

(c) Show that $\int_C f(z)\, dz = 0$ when the contour C is the unit circle $|z| = 1$ in either direction and $f(z) = \frac{z^2}{z-3}$.

(d) Show that the sequence $z_n = \frac{1}{n^3} + i$ $(n = 1, 2, 3, \ldots)$ converges to i.

3. Answer ***any three*** questions from the following : 5×3=15

(a) If z_1 and z_2 are complex numbers then show that

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

(b) Suppose a function $f(z)$ be analytic throughout a given domain D. If $|f(z)|$ is constant throughout D, then prove that $f(z)$ is constant in D.

(c) Show that the derivative of the real valued function $f(z) = |z|^2$ exists only at $z = 0$.

(d) If a function f is analytic at a given point, then prove that its derivatives of all orders are analytic there too.

(e) State Cauchy integral formula. Apply it to find $\int_C \frac{f(z)}{z+i} dz$ where $f(z) = \frac{z}{9-z^2}$ and C is the positively oriented circle $|z| = 2$.

4. Answer *either* (a) and (b) *or* (c) of the following questions : 10

(a) (i) Show that if $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$ 3

(ii) Show that the function $f(z) = e^{-y} \sin x - i e^{-y} \cos x$ is entire. 3

(b) If a function $f(z)$ is continuous and nonzero at a point z_0, then prove that $f(z) \neq 0$ throughout some neighbourhood of that point. 4

Or

(c) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε neighbourhood of a point $z_0 = x_0 + iy_0$, and suppose that

(i) the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighbourhood;

(ii) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ at (x_0, y_0).

Prove that $f'(z)$ exists and $f'(z_0) = u_x + iv_x$ where the right hand side is to be evaluated at (x_0, y_0). 10

wer **either** *(a)* and *(b)* **or** *(c)* and *(d)* of following questions : 10

Find the value of $\int_C \overline{z}\, dz$ where C is the right-hand half $z = 2e^{i\theta}\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$ of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$. 5

(b) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the 1st quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \le \frac{6\pi}{7}$$ 5

Or

(c) State Liouville's theorem. 1

(d) Prove that any polynomial

$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \quad (a_n \ne 0)$

of degree $n\,(n \ge 1)$ has at least one zero. 9

6. Answer ***either*** *(a)* and *(b)* ***or*** *(c)* and *(d)* of the following questions : 10

(a) Suppose that

$z_n = x_n + iy_n \quad (n = 1, 2, 3 \ldots)$ and

$S = X + iY$. Prove that

$$\sum_{n=1}^{\infty} z_n = S \text{ if and only if}$$

$$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$ 5

(b) Find the Maclaurin series for the entire function $f(z) = \sin z$. 5

Or

(c) Define absolutely convergent series. Prove that the absolute convergence of a series of complex numbers implies the convergence of the series. 1+3=4

(d) Find the Maclaurin series for the entire function $f(z) = \cos z$. 6